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ON THE NON-EXISTENCE OF A SUPPLEMENTARY INTEGRAL IN THE PROBLEM OF A HEAVY TWO-LINK PLANE PENDULUM*

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The non-existence of an analytic supplementary first integral in the phase variable that is independent of the energy integral is proved by the method of splitting the separatrices. The existence of certain classes of periodic solutions is proved by using Poincaré's theorem.

The question of the non-existence of an additional linear integral in the momenta is examined in /1/ in the case when a plane pendulum is comprised of two identical links. The non-existence of an additional quadratic, and therefore, linear integral in the momenta is proved in /2/ in the case when the plane mathematical pendulum is comprised of two arbitrary links.

1. We consider a two link, heavy plane pendulum that oscillates in the vertical plane. We assume that the first link rotates around a fixed horizontal axis A_1 while the second rotates around a horizontal axis A_2 coupled rigidly to the first link and is parallel to the A_1 axis.

Let G_i be the centre of mass of the i-th linkage, m_i the mass, and I_i the moment of inertia relative to the A_i axis. If $l = |A_1A_2|$, $l_1 = |A_1G_1|$, $l_2 = |A_2G_2|$, $\alpha = \angle G_1A_1A_2$, q_1, q_2 are angles formed by the segments A_1A_2, A_2G_2 with the vertical, then the expressions for the kinetic energy and the force function have the form

$$T = {}^1/_2 ((I_1 + m_2l^2) q_1^{\cdot 2} + 2m_2l_2l \cos (q_1 - q_2) q_1^{\cdot} q_2^{\cdot} + I_2q_2^{\cdot 2})$$
$$V = g (m_1l_1 \cos (q_1 + \alpha) + m_2 (l \cos q_1 + l_2 \cos q_2))$$

where g is the acceleration due to gravity.

When the conditions

$$\alpha = \pi,\ m_2l = m_1l_1 \tag{1.1}$$

are satisfied the force function V is independent of the angle q_1. If at least one of the conditions (1.1) is not satisfied, then the force function can be represented in the form

$$V = Gg \cos (q_1 + \beta) + gm_2l_2 \cos q_2$$
$$G = [(m_1l_1 \cos \alpha + m_2l)^2 + (m_1l_1 \sin \alpha)^2]^{1/2}$$
$$\cos \beta = (m_1l_1 \cos \alpha + m_2l)/G,\ \sin \beta = m_1l_1 \sin \alpha/G$$

Let $p_i = \partial T/\partial q_i$ be canonical momenta conjugate to the coordinates q_i. Then the system moiton is described by the Hamilton equations

$$q_i^{\cdot} = \partial H/\partial p_i,\ p_i^{\cdot} = -\partial H/\partial q_i,\ i = 1, 2 \tag{1.2}$$

$$H = {}^1/_2 ((I_1 + m_2l^2) I_2 - m_2^2l_2^2l^2 \cos^2 (q_1 - q_2))^{-1} \times (I_2p_1^2 - 2m_2l_2l \cos (q_1 - q_2) p_1p_2 + (I_1 + m_2l^2) p_2^2) - g (m_1l_1 \cos (q_1 + \alpha) + m_2 (l \cos q_1 + l_2 \cos q_2)) \tag{1.3}$$

2. We will examine the case when at least one of conditions (1.1) is not satisfied. We introduce the dimensionless parameter $\varepsilon_1 \geqslant 0$ into the system of equations of motion by setting $l_2 = L_2\varepsilon_1$, where $L_2 > 0$ is a constant with the dimensions of length.

The Hamiltonian (1.3) is an analytic function of the momenta p_i, the coordinates q_i and the parameter $\varepsilon_1 \in [0, [(I_1 + m_2l^2) I_2 (m_2l_2l)^{-2}]^{1/2})$. Its series expansion in powers of the parameter ε_1 has the form

$$H (p_1, p_2, q_1, q_2) = H_0 (p_1, p_2, q_1) + \varepsilon_1H_1 (p_1, p_2, q_1, q_2) + \ldots \tag{2.1}$$

$$H_0 = {}^1/_2 (a_1p_1^2 + a_2p_2^2) - Gg \cos (q_1 + \beta) \tag{2.2}$$

$$H_1 = -a_0^{-1}a_1a_2p_1p_2 \cos (q_1 - q_2) - gm_2L_2 \cos q_2 \tag{2.3}$$
$$a_0 = (m_2L_2l)^{-1},\ a_1 = (I_1 + m_2l^2)^{-1},\ a_2 = I_2^{-1}.$$

For $\varepsilon_1 = 0$ the system equations of motion (1.2) with the Hamilton function (2.1) is Liouville integrable: in addition to the energy integral $F_1 = H_0$ it possesses the integral $F_2 = p_2$ corresponding to the cyclic coordinate q_2. In this case the first link moves as a physical pendulum while the second link performs uniform motion around the A_2 axis with angular velocity $\omega_2 = P_2/I_2$ for $F_2 = P_2 \neq 0$.

For $\varepsilon_1 = 0$ system (1.2) possesses two particular periodic solutions $(P_2 \neq 0)$

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$$x_\pi(t, 0) = \{p_1 = 0,\ p_2 = P_2,\ q_1 = \pi - \beta,\ q_2 = \omega_2 t + q_{20}\}$$
$$x_0(t, 0) = \{p_1 = 0,\ p_2 = P_2,\ q_1 = -\beta,\ q_2 = \omega_2 t + q_{20}\}$$

of period $T_2 = 2\pi\omega_2^{-1}$ located on the energy integral levels $h_\pi = {}^1/_2 I_2\omega_2^2 + Gg$ and $h_0 = {}^1/_2 I_2\omega_2^2 - Gg$, respectively. Let us clarify whether for sufficiently small values of ε_1 there exist one-parameter families of periodic solutions analytically dependent on the parameter ε_1 for system (1.2), and located at the energy integral levels $\{H = h_\pi\}$ and $\{H = h_0\}$ and going over into the solutions $x_\pi(t, 0)$ and $x_0(t, 0)$ for $\varepsilon_1 = 0$.

Let

$$Z_\sigma(T_2) = \begin{Vmatrix} X_\sigma(T_2) & f_\sigma \\ \Psi_\sigma & 0 \end{Vmatrix}$$

where $X_\sigma(T_2)$ is the monodromy matrix of the periodic solution $x_\sigma(t, 0)$, $\sigma = 0, \pi$

$$f_\sigma = \operatorname{col}(-\partial H_0/\partial q_1,\ \partial H_0/\partial p_1,\ -\partial H_0/\partial q_2,\ \partial H_0/\partial p_2)_{x_\sigma(T_2, 0)}$$
$$\psi_\sigma = (\partial H_0/\partial p_1,\ \partial H_0/\partial q_1,\ \partial H_0/\partial p_2,\ \partial H_0/\partial q_2)_{x_\sigma(T_2, 0)}$$

Following Poincaré's theorem on periodic solutions of systems admitting of first integrals /3, 4/, we calculate the rank of the matrix $Z_\sigma(T_2)$. The rank of the matrix $Z_\pi(T_0)$ equals four: the minor

$$M_{34} = 2\omega_2^2(1 - \operatorname{ch}(2\pi\lambda_1\omega_2^{-1})),\quad \lambda_1 = (a_1 Gg)^{1/2}$$

differs from zero. Therefore, according to Poincaré's theorem, for sufficiently small values of ε_1 at the energy integral level $\{H = h_\pi\}$ a one-parameter family of periodic solutions $x_\pi(t, \varepsilon_1)$ exists that is analytically dependent on the parameter ε_1 and reduces to $x_\pi(t, 0)$ when $\varepsilon_1 = 0$.

Then rank of the matrix $Z_0(T_2)$ is not greater than four: the minor

$$M_{34} = 2\omega_2^2(1 - \cos(2\pi\lambda_1\omega_2^{-1}))$$

differs from zero if

$$\lambda_1 \neq \omega_2 k,\quad k = 0, \pm 1, \ldots \tag{2.4}$$

In this case the rank $Z_0(T_2) = 4$ and by Poincaré's theorem for sufficiently small values of ε_1 a one-parameter family of periodic solutions $x_0(t, \varepsilon_1)$ exists located at the energy integral level $\{H = h_0\}$ and reducing to $x_0(t, 0)$ when $\varepsilon_1 = 0$.

Remark. If conditions (2.4) are not satisfied, then the existence of periodic solutions close to $x_0(t, 0)$ for small $\varepsilon_1 \neq 0$ follows from the Kolmogorov-Arnol'd-Moser theory. However, the question of whether these solutions form a family dependent analytically on ε_1 requires additional investigation.

3. We will examine the one-parameter family of periodic solutions $x_\pi(t, \varepsilon_1)$. The solution $x_\pi(t, 0)$ is an unstable periodic solution of hyperbolic type. Therefore, for sufficiently small values of ε_1 the periodic solutions $x_\pi(t, \varepsilon_1)$ are also hyperbolic. For these solutions separatrices exist, i.e. two two-dimensional invariant asymptotic surfaces

$$\Lambda_u(\varepsilon_1) = \Lambda_u^+(\varepsilon_1) \cup \{x_\pi(t, \varepsilon_1)\} \cup \Lambda_u^-(\varepsilon_1)$$
$$\Lambda_s(\varepsilon_1) = \Lambda_s^+(\varepsilon_1) \cup \{x_\pi(t, \varepsilon_1)\} \cup \Lambda_s^-(\varepsilon_1)$$

filled compactly with trajectories approaching $x_\pi(t, \varepsilon_1)$ asymptotically as $t \to \mp\infty$.

For $\varepsilon_1 = 0$ the branches of the separatrices $\Lambda_u^+(0)$ and $\Lambda_s^+(0)$, $\Lambda_u^-(0)$ and $\Lambda_s^-(0)$ coincide and consist of the solutions $x_a^\pm(t, q_{20})$

$$\sin q_1^\pm = \pm 2\operatorname{sh}\tau_1/\operatorname{ch}^2\tau_1,\quad \cos q_1 = 2/\operatorname{ch}^2\tau_1 - 1$$
$$p_1^\pm = \pm 2a_1^{-1}\lambda_1/\operatorname{ch}\tau_1,\quad q_2 = \gamma_1\tau_1 + q_{20},\quad p_2 = P_2$$
$$\tau_1 = \lambda_1 t,\quad \gamma_1 = \lambda_1^{-1}\omega_2$$

Theorem 1. If at least one of the conditions (1.1) is not satisfied, then for sufficiently small values of $\varepsilon_1 \neq 0$ the branches of the separatrices $\Lambda_u^+(\varepsilon_1)$ and $\Lambda_s^+(\varepsilon_1)$, $\Lambda_u^-(\varepsilon_1)$ and $\Lambda_s^-(\varepsilon_1)$ intersect transversely and the system of Eqs.(1.2) has no supplementary first integral analytic in the phase variables.

Proof. According to (2.3), the function H_1 has the form

$$H_1 = h_1^*\exp(iq_2) + h_{-1}^*\exp(-iq_2) = h_1 + h_{-1}$$

where

$$h_{\pm 1}^* = -{}^1/_2(a_0^{-1}a_1a_2p_1p_2\exp(\mp iq_1) + m_2L_2g)$$

Following /5/, we find the functions

$$J^\pm(q_2) = \sum_k J_k^\pm \exp(ikq_2)$$
$$J_k^\pm = -2\pi k\left(1 - \exp\left(-\frac{2\pi k}{\lambda_1}\frac{\partial H_0}{\partial P_2}\right)\right)\sum_{\Pi_1}\operatorname{res} h_k(z_a^\pm(t))$$

Here $z_a^\pm(t)$ is the analytical continuation of the solution $x_a^\pm(t, 0)$ in the strip

$$\Pi_1\colon\ 0 \leqslant \operatorname{Im} t < 2\pi/\lambda_1$$

Evaluating the coefficients $J_k^\pm$ using residues, we obtain

$$J^{\pm}(q_2) = 2\pi a_0^{-1}\omega_2^2\gamma_1^{-1}\left(\frac{1}{\operatorname{ch}(\pi\gamma_1/2)} \pm \frac{1}{\operatorname{sh}(\pi\gamma_1/2)}\right)\sin(q_2+\beta)$$

Since the functions $J^{\pm}(q_2)$ have isolated zeros, then according to /5/ (Theorem 1), for sufficiently small values of $\varepsilon_1 \neq 0$ the pairs of separatrix branches $\Lambda_u^+(\varepsilon_1)$ and $\Lambda_s^+(\varepsilon_1)$, $\Lambda_u^-(\varepsilon_1)$ and $\Lambda_s^-(\varepsilon_1)$ split and intersect transversely, while the system of equations of motion has no additional first integral analytic in the phase variables.

4. We consider the case when both conditions (1.1) are satisfied. We introduce the dimensionless parameter $\varepsilon_2 \geqslant 0$ into the system of equations of motion by setting $l = L_0\varepsilon_2$, $l_1 = L_1\varepsilon_2$, where $L_0 > 0$, $L_1 > 0$ are constants with the dimensions of length and satisfying the condition $m_1L_1 = m_2L_0$ by virtue of the second relationship in (1.1).

The Hamilton function (1.3) is analytic in the phase variables p_i, q_i and the parameter $\varepsilon_2 \in [0, (I_1m_2^{-1}L_0^{-2})^{1/2})$, its power series expansion in ε_2 has the form

$$H(p_1, p_2, q_1, q_2) = H_0(p_1, p_2, q_2) + \varepsilon_2 H_1(p_1, p_2, q_1, q_2) + \ldots$$

$$H_0 = {}^1/_2(I_1^{-1}p_1^2 + I_2^{-1}p_2^2) - m_2l_2g\cos q_2 \tag{4.1}$$

$$H_1 = -m_2l_2L_0\cos(q_1 - q_2)\,p_1p_2 \tag{4.2}$$

For $\varepsilon_2 = 0$ the system of Eqs.(1.2) with the Hamiltonian (4.1) is completely integrable: in addition to the energy integral $\Phi_0 = H_0$ it possesses the integral $\Phi_1 = p_1$ corresponding to the cyclic coordinate q_1. In this case for $\Phi_1 = P_1 \neq 0$ the first linkage performs uniform rotations around the A_1 axis with angular velocity $\omega_1 = P_1/I_1$ while the second linkage oscillates as a physical pendulum.

At energy integral levels

$$\eta_\pi = I_1\omega_1^2/2 + m_2l_2g,\ \eta_0 = I_1\omega_1^2/2 - m_2l_2g$$

where $\varepsilon_2 = 0$, periodic solutions exist $(P_1 \neq 0)$

$$y_\pi(t, 0) = \{p_1 = P_1,\ p_2 = 0,\ q_1 = \omega_1t + q_{10},\ q_2 = \pi\}$$

$$y_0(t, 0) = \{p_1 = P_1,\ p_2 = 0,\ q_1 = \omega_1t + q_{10},\ q_2 = 0\}$$

of period $T_1 = 2\pi\omega_1^{-1}$. According to Poincaré's theorem, the solution $y_\pi(t, 0)$ here belongs to a family of periodic solutions of hyperbolic type $\{y_\pi(t, \varepsilon_2)\}$ that is analytically dependent on the small parameter ε_2 and located on $\{H = \eta_\pi\}$. When the conditions

$$(m_2l_2g/I_2)^{1/2} \neq \omega_1k,\ k = 0, \pm 1, \ldots$$

are satisfied the solution $y_0(t, 0)$ also belongs to the family of periodic soltuions $\{y_\pi(t, \varepsilon_2)\}$ located on $\{H = \eta_0\}$ that depend analytically on the small parameter ε_2.

For sufficiently small values of ε_2 the periodic solutions $y_\pi(t, \varepsilon_2)$ also possess invariant asymptotic surfaces, separatrices, where the following holds.

Theorem 2. If conditions (1.1) are satisfied, then for sufficiently small values of $\varepsilon_2 \neq 0$ the separatrix branches of the hyperbolic periodic solution $y_\pi(t, \varepsilon_2)$ intersect transversely and Eqs.(1.2) have no supplementary first integral analytic in the phase variables.

The proof of Theorem 2 is analogous to the proof of Theorem 1.

The lack of a supplementary integral for the equations of motion of a two-link pendulum in the general case enables us to clarify the nature of the complex motion of this mechanical system.

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